Straight Canted Dagger Boards and Curved Dagger Boards

Cris Bolle
August 2008

This paper calculates the basic lift and drag equations for straight canted and curved (banana) bagger boards under the assumption of high aspect ratio and negligible tip vortex. The equivalence between straight canted and curved boards is calculated.

We will start by defining the geometry. Let's assume we have a straight board with a symmetric profile that sits vertically in the water. The best way to define the orientation of the board is by a vector $\hat{n}$ normal to the board's center plane. In Figure 1 the board points along the X axis and the vector $\hat{n}$ points along the Y axis. If we add a Z axis that points out of the page, the unitary vector $\hat{n}$ will be given by:

$$\hat{n} = (0,1,0)$$

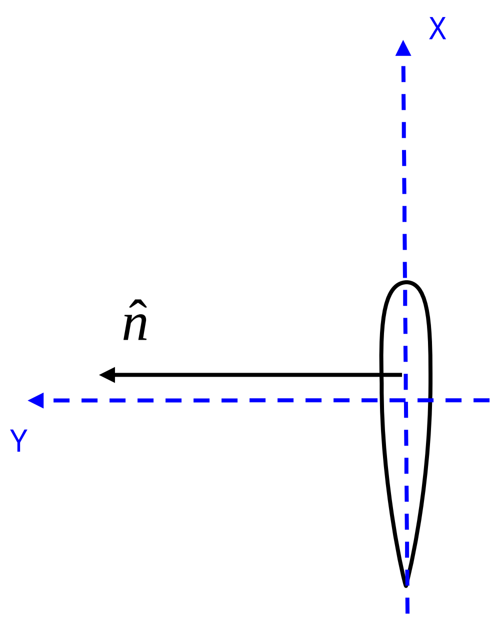

Figure 1: The board orientation is given by the vector normal to its center plane.

Now, since everybody is so excited about canted boards, let's rotate this board around the X axis by an angle β. In Figure 2 we show this board as seen along the X axis, or more precisely, if we look in the +X direction. The angle β_S (S for straight board) is zero for a vertical board and π/2 radians or 90 degree if the board is completely horizontal. Now the vector that defines the orientation of the board takes on a slightly more complicated form, that is:

$$\hat{n} = (0, \cos(\beta_S), \sin(\beta_S))$$

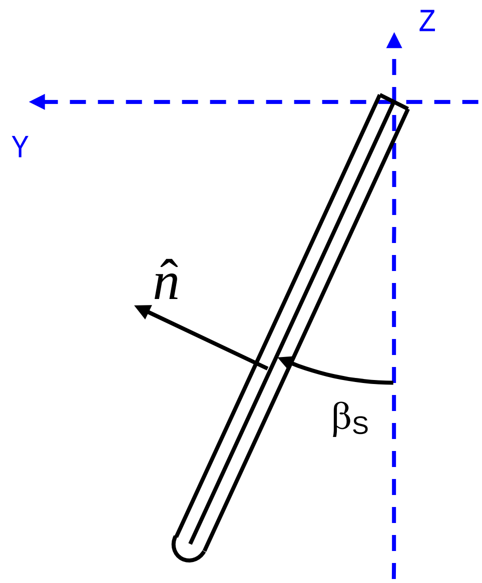

Figure 2: Back view of a canted board

Since we rotated the board around the X axis, the vector sits completely in the Y-Z plane. The sine and cosine come from our requirement for $\hat{n}$ to be a unit vector.

Now that we have a canted board, all we need to add is the difference between the heading and the course.

Figure 3 shows a top view of the new geometry. The boat axis and the board point slightly to the left of the X axis to generate some lift. The angle α between the boat axis (heading) and the actual motion (course) determine the angle of attack for the foil. But beware; the angle of attack is not equal to α for a canted board.

The next step is to take the board of figure 2 and rotate it around the Z axis by α. The rotation matrix in this case is rather simple:

$$R_Z(\alpha) = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The rotated vector $\hat{n}$ is given by:

$$\hat{n}(\alpha, \beta_S) = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \cos(\beta_S) \\ \sin(\beta_S) \end{bmatrix}$$

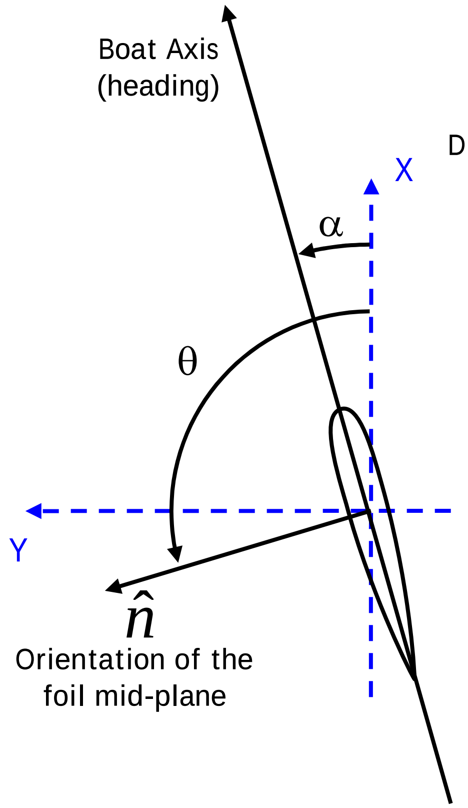

Figure 3: Top view of the geometry in the Z=0 plane.

Or, after performing the matrix product:

$$\hat{n}(\alpha, \beta_S) = \left(-\sin(\alpha)\cos(\beta_S), \cos(\alpha)\cos(\beta_S), \sin(\beta_S)\right)$$

You can see that $\hat{n}(\alpha = 0, \beta_S) = (0, \cos(\beta_S), \sin(\beta_S))$ as before and that for $\beta_S = \pi / 2$ (the board is tilted completely to a horizontal position) $\hat{n}$ is independent of α as should be.

Now that we have $\hat{n}(\alpha, \beta_S)$ (any heading of the boat, any canting of the board) what we would like to calculate is the actual angle of attack. That angle will determine the lift. This lift we will be able to separate into two components, the sideways and upwards.

The actual angle of attack we will call γ. We can get it from the scalar product between the course (the direction of motion of the water) and the board orientation $\hat{n}(\alpha, \beta_S)$.

$$\hat{x} \bullet \hat{n} = |\hat{x}||\hat{n}|\cos(\theta)$$

Since $\hat{x} = (1,0,0)$ and both vectors are unitary the scalar product is simply:

$$\hat{x} \bullet \hat{n} = \cos(\theta) = -\sin(\alpha)\cos(\beta_S)$$

The angle of attack γ will be the angle θ minus 90 degree ($\pi/2$ radians):

$$\gamma = \theta - \pi/2 = \arccos(-\sin(\alpha)\cos(\beta_S)) - \pi/2$$

As a sanity check we can see that γ will be zero if the heading matches the course ($\alpha = 0$ or no lift) independently of the value of the canting β_S. Similarly we can also see that the angle of attack will be zero for a horizontal board or $\beta_S = \pi/2$ independently of the heading and course orientation.

Since $\arcsin(x) = \pi/2 - \arccos(x)$ the last expression can be rewritten as:

$$\gamma = \arcsin(-\sin(\alpha)\cos(\beta_S))$$

Now comes the first approximation and I did check this one graphically and it works like a charm. Since the difference α between heading and course is small we can approximate:

$$\sin(\alpha) \cong \alpha$$
$$\arcsin(\alpha) \cong \alpha$$

Then the angle of attack is:

$$\gamma = -\alpha\cos(\beta_S)$$

The minus sign can be ignored. We could actually almost have guessed this result since the angle of attack has to equal α for a vertical board ($\beta_S = 0$) and has to be zero for a horizontal board.

We would like to know what the lift components are for a straight board, for a curved banana board and also find the equivalence between both types of boards.

Lift is orthogonal to the direction of motion and must therefore lie in the Y-Z plane. If $l(\gamma)$ denotes the lift per unit area of a board then the two components, the side force and upward force, are given by:

$$l_Y(\gamma) = l(\gamma)\cos(\beta_S)$$
$$l_Z(\gamma) = l(\gamma)\sin(\beta_S)$$

Here comes the second approximation. In principle, for a fixed angle of attack, the lift is not uniform along the board and it will fall off toward the tip due to the tip vortex. Now, since we talk about high aspect ratio boards and we want to compare similar straight and curved boards, the lift is fairly uniform and only falls off close to the tip. We will therefore ignore all that since I don't have at this point a nice expression for that contribution and we may still learn something from this exercise.

For a straight board of length *H* and cord *c* the total lift is simply:

$$L_Y(\gamma) = l(\gamma)Hc\cos(\beta_S)$$
$$L_Z(\gamma) = l(\gamma)Hc\sin(\beta_S)$$

The lift force for per unit area for a symmetric board profile is:

$$l = \tfrac{1}{2}\rho v^2 C_L$$

The fluid density is ρ, the boat speed is v and the lift coefficient C_L.

Since the lift coefficient for a symmetric board varies linearly with angle and with zero offset, it can be written as $C_L = k\gamma$ or $C_L = k\alpha\cos(\beta_S)$. The two components of the lift are therefore:

$$L_Y = \tfrac{1}{2}\rho v^2 Hck\alpha\cos^2(\beta_S) \qquad (1)$$
$$L_Z = \tfrac{1}{2}\rho v^2 Hck\alpha\sin(\beta_S)\cos(\beta_S) \qquad (2)$$

For a curved board we will refer to Figure 4. The length H is now along the curved board. We will assume the board starts vertically and ends at an angle β_C. The radius of curvature of the board is *R*. Now the two components of the lift change along the board and we need to integrate the lift component per unit length between zero and *H*.

$$L_Y(\gamma) = \int_0^H l_Y(\gamma)dh$$

We'll turn the integration in *H* to an integration in angle by using:

$$H = \beta_C R$$
$$dh = R\, d\beta$$

The integral for the sideways force turns into:

$$L_Y(\gamma) = R\int_0^{\beta_C} l_Y(\gamma)d\beta$$

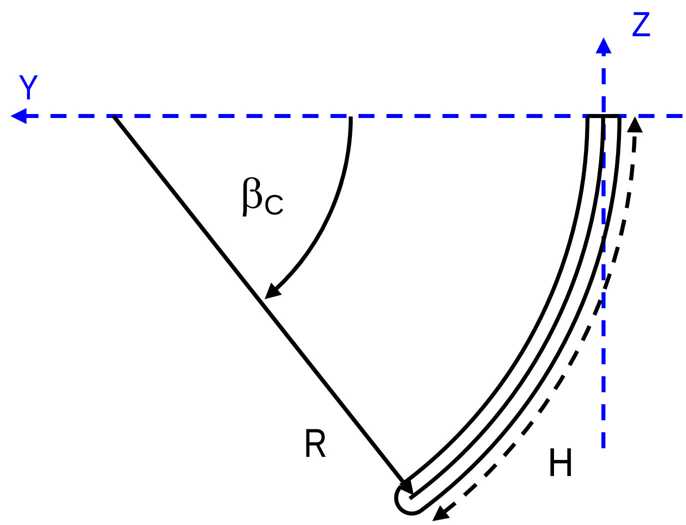

Figure 4: Curved dagger board geometry

The integral of the lift is then:

$$L_Y = \tfrac{1}{2}\rho v^2 Rck\alpha \int_0^{\beta_C} \cos^2(\beta)d\beta$$

This integral is:

$$L_Y = \tfrac{1}{2}\rho v^2 Rck\alpha \left[\tfrac{1}{2}\beta + \tfrac{1}{4}\sin(2\beta)\right]_0^{\beta_C}$$

$$L_Y = \tfrac{1}{2}\rho v^2 Rck\alpha \left[\tfrac{1}{2}\beta_C + \tfrac{1}{4}\sin(2\beta_C)\right]$$

By using the relation $A = H_C c = Rc\beta_C$.

$$L_Y = \tfrac{1}{2}\rho v^2 Ak\alpha \left[\tfrac{1}{2} + \tfrac{1}{4}\frac{\sin(2\beta_C)}{\beta_C}\right] \qquad (3)$$

In a similar way we can calculate the vertical component of the lift:

$$L_Z = \tfrac{1}{2}\rho v^2 Rck\alpha \int_0^{\beta_C} \sin(\beta)\cos(\beta)d\beta$$

$$L_Z = \tfrac{1}{2}\rho v^2 Rck\alpha \left[\tfrac{1}{2}\sin^2(\beta)\right]_0^{\beta_C}$$

$$L_Z = \tfrac{1}{2}\rho v^2 Rck\alpha \ \tfrac{1}{2}\sin^2(\beta_C)$$

$$L_Z = \tfrac{1}{2}\rho v^2 Ak\alpha \frac{\sin^2(\beta_C)}{2\beta_C} \qquad (4)$$

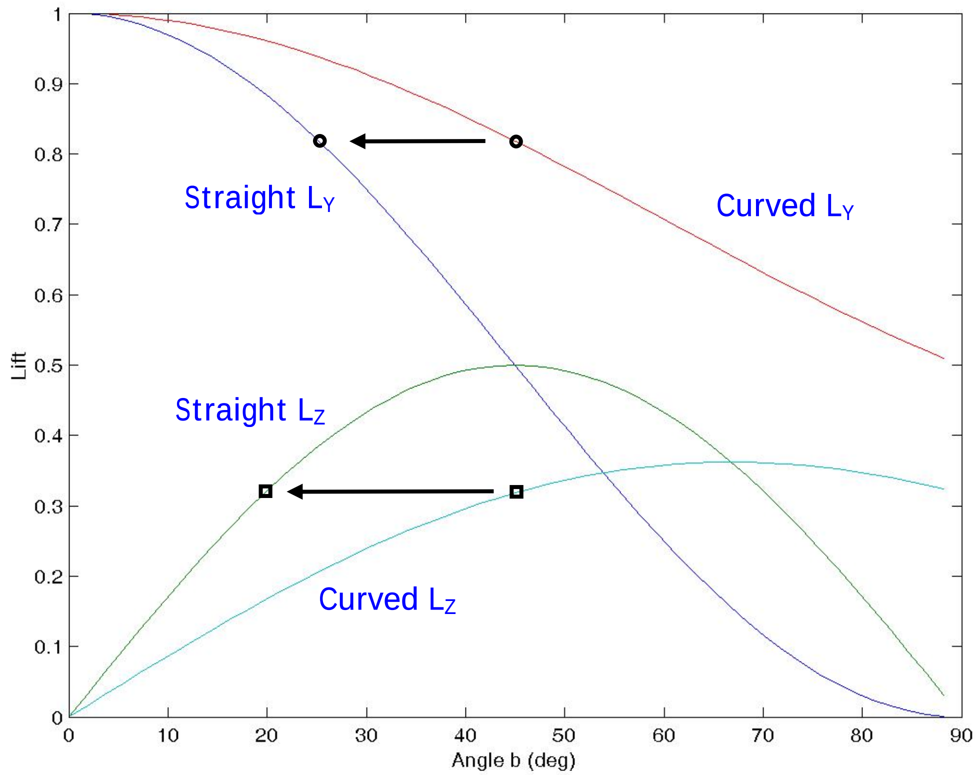

Figure 5: Lift components for boards of equal length as a function of canting angle or curve span.

As a sanity check it is easy to see that for a constant board length H and $\beta_C \to 0$ you recover the straight board result for $\beta_S = 0$. You can also see that in equations (1) to (4) we can separate the common lift contribution $\frac{1}{2}\rho v^2 Ak\alpha$ from the geometric factors like $\frac{\sin^2(\beta_C)}{2\beta_C}$.

The question now that we have equations 1 to 4 is if a curved board of length H and radius R (or angle β_C) is equivalent to some straight board of length H and canting angle β_S. The short answer is that there is no exact equivalence since we are restricting the set

of equations by asking for both boards to be equally long. Figure 5 shows the geometric part of the components of the lift as a function of angle, β_S for the straight board or β_C for the curved board. It's easy to see that there isn't a simple correspondence between the components for the two types of boards. A curved board that spans 45 degree would be matched in the side lift by a straight board canted 25 degree (circles) but the vertical lift would be matched by a straight board canted only 20 degree (squares). The correspondence is never the less fairly close.

To find the canted board that is exactly equivalent to a curved board we need to leave both the lengths and angles as independent variables and solve the set of equations. The starting equations for the straight board are:

$$L_Y = \tfrac{1}{2}\rho v^2 H_S ck\alpha \cos^2(\beta_S)$$
$$L_Z = \tfrac{1}{2}\rho v^2 H_S ck\alpha \sin(\beta_S)\cos(\beta_S)$$

And for the curved boards the starting equations are:

$$L_Y = \tfrac{1}{2}\rho v^2 H_C ck\alpha \left[\tfrac{1}{2} + \tfrac{1}{4}\frac{\sin(2\beta_C)}{\beta_C}\right]$$
$$L_Z = \tfrac{1}{2}\rho v^2 H_C ck\alpha \frac{\sin^2(\beta_C)}{2\beta_C}$$

Where we used the same cord for both boards and used $A_S = H_S c$ and similarly $A_C = H_C c$ for the curved board. Equating L_Y and L_Z from both sets of equations and canceling all the common factors we get:

$$H_S \cos^2(\beta_S) = H_C\left[\tfrac{1}{2} + \tfrac{1}{4}\frac{\sin(2\beta_C)}{\beta_C}\right]$$
$$H_S \sin(\beta_S)\cos(\beta_S) = H_C \frac{\sin^2(\beta_C)}{2\beta_C}$$

By equating H_S from both equations and cleaning up we get for the equivalent canting angle of a straight board:

$$\tan(\beta_S) = \frac{\sin^2(\beta_C)}{\beta_C + \tfrac{1}{2}\sin(2\beta_C)} \qquad (5)$$

This ugly expression can be replaced in any of the two previous equations to obtain H_S but it is easier just to replace the number for β_S in the equation for H_S:

$$H_S = \tfrac{1}{2}\frac{H_C}{\beta_C}\frac{\sin^2(\beta_C)}{\sin(\beta_S)\cos(\beta_S)} \qquad (6)$$

When evaluating these equations always remember to use angles measured in radians or you will get some strange results! We can evaluate these equations for the case of a board that spans 45 degree and is 1.2 meter long. The equivalent straight board would be tilted at 21.25 degree and would be 1.13 meter long. To get an idea of what all this looks like we will plot H_S / H_C and β_S as a function of β_C. From Figure 5 you see that the straight board will be tilted at an angle that is roughly ½ the span of the curved board. The straight board will be always shorter than the curved board.

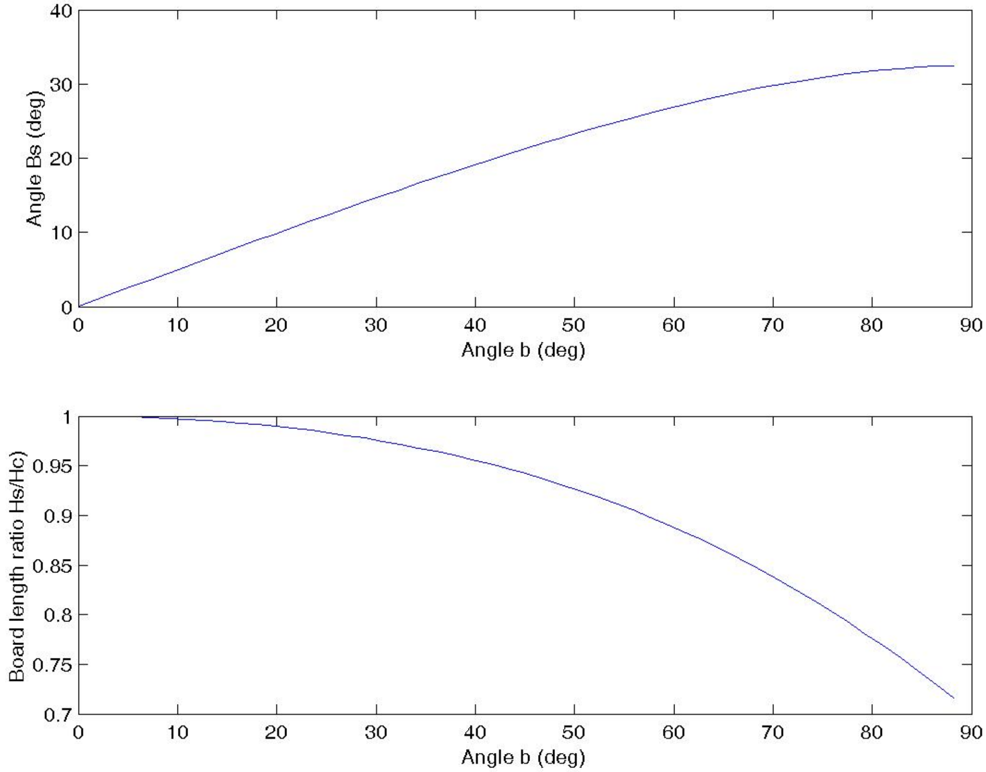

Figure 6: Angle β_S as a function of β_C (top) length ratio H_S / H_C as a function of β_C (bottom)

If the curved board does not exit the hull vertically but with an angle β_0 the equations can still be used but β_0 has to be subtracted from β_C and later added to β_S .

Another important question is how the lift changes as the board is inserted. In this case *R* is constant while β_C changes between zero (the board is flush with the hull bottom) and some angle (β_C in Figure 4). The equations are:

$$L_Y = \tfrac{1}{2}\rho v^2 Rck\alpha\left[\tfrac{1}{2}\beta_C + \tfrac{1}{4}\sin(2\beta_C)\right]$$

$$L_Z = \tfrac{1}{2}\rho v^2 Rck\alpha \frac{\sin^2(\beta_C)}{2}$$

If we leave the common factor $\tfrac{1}{2}\rho v^2 Rck\alpha$ out we will see that initially the side lift increases linearly while the vertical lift is very small; as expected since there is not much curve to it. As the board is further inserted the vertical lift starts to kick in. Figure 7 shows both components. For a straight board both components are always proportional to the expose board length and will follow a straight line.

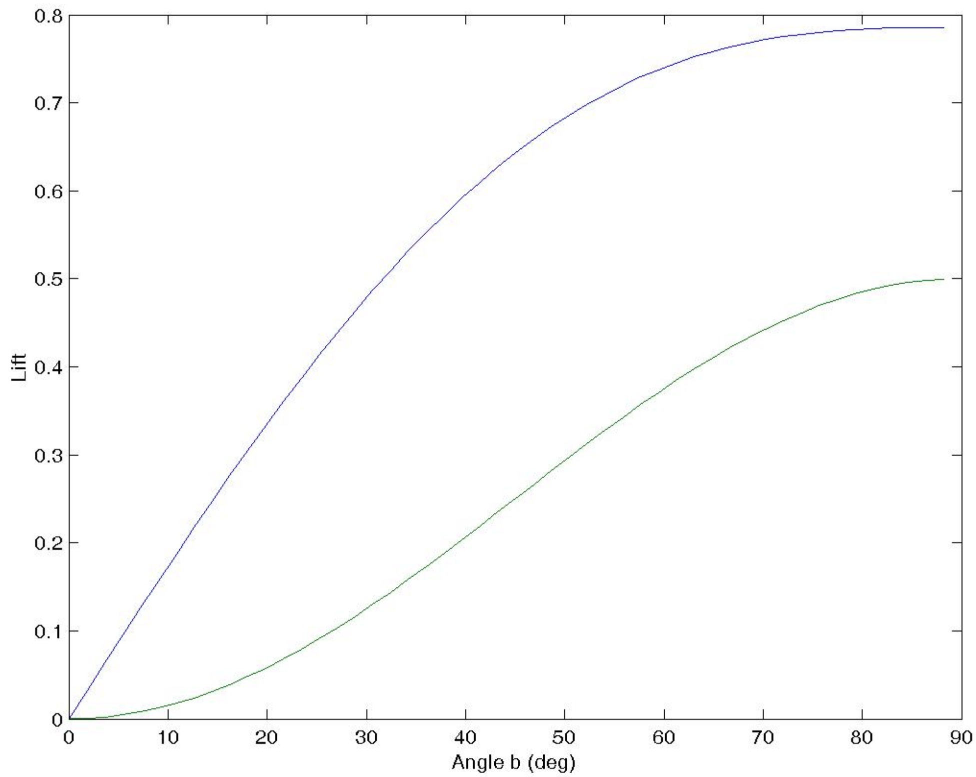

Figure 7: Side lift (top curve) and vertical lift (bottom curve) for a board of fixed radius of curvature as it is inserted.

At this point it helps to have a little discussion and find the meaning of all these equations. Let’s start with the straight board.

If the board sits vertically at an angle of attack $\gamma = \alpha$ it generates a certain side force and no vertical force. As we tilt the board by an angle β_S (we keep the same course α) the side force will decrease by $\cos^2(\beta_S)$ and the vertical force will increase by $\sin(\beta_S)\cos(\beta_S)$. So if the board is tilted inward by 15 degrees you will have lost 6.7% of the original ($\beta_S = 0$) side force but you will have a vertical force that is 25% of that original side force. If the original angle of attack was five degree now it dropped to $\gamma = 4.83$ degree.

If we would like to recover the original side force we have to increase α by 6.7% or increase the board size by 6.7%.

If we tilt the board to 45 degree (see Figure 5) the side force will have dropped to ½ of it’s original value buy we will have the maximum possible vertical force, also equal to ½ of the original side force value.
But now we would like to recover the original side force. We can do this by doubling α or by doubling the area of the board. This in turn will increase the board drag.

Following equation (5) and (6) or Figure 6, canted straight boards and curved boards are perfectly equivalent.

The next step is to look at the drag coming from the board.

The drag force for a straight board of uniform cross section and high aspect ratio (we neglect the tip vortex effect) is:

$$F_D = \tfrac{1}{2}\rho v^2 C_D A$$

And the drag coefficient for a symmetric foil without drag bucket is quadratic in the angle of attack with a positive offset:

$$C_D = k_1 + k_2\gamma^2$$

Since the board area is the length H times the cord c and $\gamma = \alpha\cos(\beta)$:

$$F_D = \tfrac{1}{2}\rho v^2 cH(k_1 + k_2\alpha^2\cos^2(\beta_S))$$

This drag force has only one component and that is $-\hat{x}$ since it goes against the direction of motion.

For the curved board we have to perform an integral along the curve:

$$F_D = \int_0^H f_D dh$$

Where f_D is the drag force per unit length or:

$$f_D = \tfrac{1}{2}\rho v^2 c(k_1 + k_2\alpha^2\cos^2(\beta))$$

$$F_D = \tfrac{1}{2}\rho v^2 c\int_0^H (k_1 + k_2\alpha^2\cos^2(\beta))dh$$

$$F_D = \tfrac{1}{2}\rho v^2 cR\int_0^{\beta_0} (k_1 + k_2\alpha^2\cos^2(\beta))d\beta$$

This integral is similar to our previous lift integral and yields:

$$F_D = \tfrac{1}{2}\rho v^2 cR[k_1\beta_C + k_2\alpha^2(\tfrac{1}{2}\beta_C + \tfrac{1}{4}\sin(2\beta_C))]$$

$$F_D = \tfrac{1}{2}\rho v^2 A[k_1 + k_2\alpha^2(\tfrac{1}{2} + \tfrac{1}{4}\frac{\sin(2\beta_C)}{\beta_C})] \qquad (7)$$

At this point we will have to put some real numbers into the equations to see which board gives you higher drag. The right thing here would be to fire up XFoil and get some numbers for a specific foil. Since that is too much work at this point we will just take some numbers I found online. For a NACA 0010 at $\mathrm{Re} = 5e5$ I found $k_1 = 0.008$ and $k_2 = 1.11e-4\frac{1}{Deg^2}$. These values assume angles measured in degree. Since in our case α has to be measured in radian, it would be best to switch these values to radian and avoid some mistakes:

$$k_1 = 0.008$$
$$k_2 = 0.364$$

For the curved board from our previous example we had $H_C = 1.2m$ or $R = 1.528m$, $\beta_C = 0.7854$ radian (45 degree) and let's take a cord $c = 0.17m$. For the equivalent straight board we found $\beta_S = 0.3709$ radian (21.25 degree), $H_S = 1.13m$ and the same cord. We can actually leave the cord out since it is a common factor for both equations along with the fluid density and the speed. We can also replace $H_C = R\beta_C$. If we do all that we will find that the drag of the curved board is larger than the drag of the straight board by H_C / H_S or 6%.

The conclusion is that every curved board has an equivalent straight board with roughly half the angle and a bit shorter. This reduced length leads to a smaller drag that is negligible in the big picture since it is much smaller than the hull drag.

Vertical lift always comes at the expense of increased dagger board drag. The next question is if it is offset by an even larger reduction in hull drag.

Also, if the board is canted forward (a rotation around the Y axis) the vertical lift will be enhanced and a non zero lift will result even for $\alpha = 0$!